

14.7 Local Max/Min

Consider the surface $z = f(x,y)$.

Some Terminology:

A **local maximum** occurs at (a,b) if $f(a,b)$ is larger than *all* values "near" it (top of a hill).

A **local minimum** occurs at (a,b) if $f(a,b)$ is smaller than *all* values "near" it (bottom of a valley).

A **critical point** is a point (a,b) where **BOTH**

$f_x(a,b) = 0$ AND $f_y(a,b) = 0$ or where either partial doesn't exist.

If $f_x(a,b) = 0$ and $f_y(a,b) = 0$, and (a,b) is not a local max or min, then we call it a **saddle point**.

Example: Find the critical points of

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

$$\boxed{1} \quad f_x = 3y + 6x^2 + 9x \stackrel{?}{=} 0$$

$$\boxed{2} \quad f_y = 3x - y \stackrel{?}{=} 0$$

REMINDER OF HOW TO SOLVE A SYSTEM OF EQUATIONS.

STEP 1 Solve for one variable in one equation (whichever is easier)

$$\boxed{2} \Rightarrow 3x = y$$

STEP 2 Substitute into other equation

$$\boxed{2} \& \boxed{1} \Rightarrow 3(3x) + 6x^2 + 9x \stackrel{?}{=} 0$$
$$6x^2 + 18x = 0$$

STEP 3 Solve!

$$6x(x+3) = 0$$
$$x=0 \quad \text{or} \quad x+3=0$$
$$x=-3$$

Two Answers

$$x=0 \Rightarrow y=3(0)=0 \Rightarrow (x,y) = (0,0)$$

$$x=-3 \Rightarrow y=3(-3)=-9 \Rightarrow (x,y) = (-3,-9)$$

CHECK!

Example: Find and classify all critical points for

$$f(x, y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

$$f_{xx} = 12x + 9$$

$$f_{yy} = -1$$

$$f_{xy} = 3 = f_{yx}$$

$$D = f_{xx} f_{yy} - \underbrace{f_{xy} f_{yx}}_{(f_{xy})^2} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$D = (12x + 9)(-1) - (3)^2$$

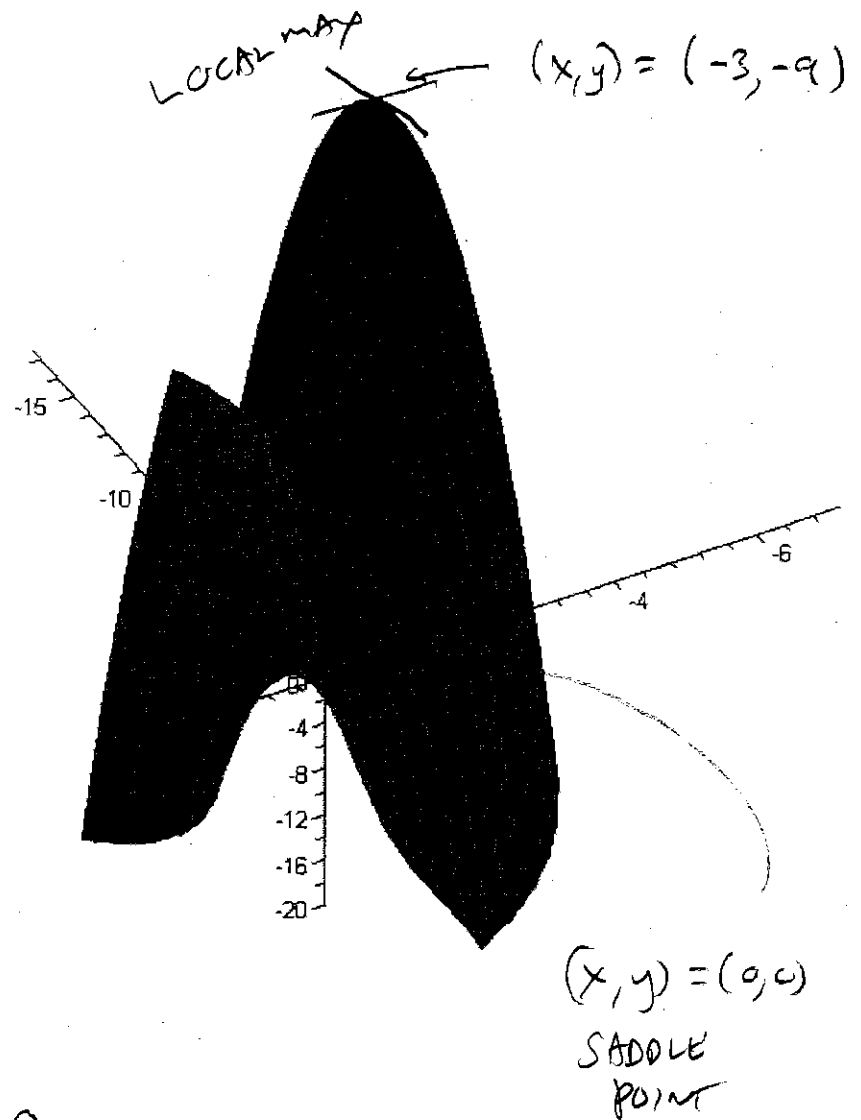
$$(0, 0) \Rightarrow D = (9)(-1) - 9 = -18 < 0$$

$$(-3, -9) \Rightarrow D = \underbrace{(-36 + 9)}_{-27} \underbrace{(-1)}_{-1} - 9 = 27 - 9 = 18 > 0$$

ASIDE: LOCAL MAX VALUE = OUTPUT = $f(-3, -9) =$

$$3(-3)(-9) - \frac{1}{2}(-9)^2 + 2(-3)^3 + \frac{9}{2}(-3)^2$$

$$= \boxed{27} = \text{LOCAL MAX}$$



Second Derivative Test

Let (a,b) be a critical point.

Find all **second** partials at (a,b)

and compute

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

1. If $D > 0$, then the concavity is the same in all directions. So

(a) If $f_{xx} > 0$, then it is concave up in all directions. **Local Minimum.** $\Rightarrow f_{yy} > 0$ As well

(b) If $f_{xx} < 0$, then it is concave down in all directions. **Local Maximum.** $\Rightarrow f_{xx} < 0$ As well

2. If $D < 0$, then the concavity changes in some direction. **Saddle Point.**

3. If $D = 0$, the test is **inconclusive**.
(need a contour map)

Quick Examples:

1. $f(x,y) = 15 - x^2 - y^2$,

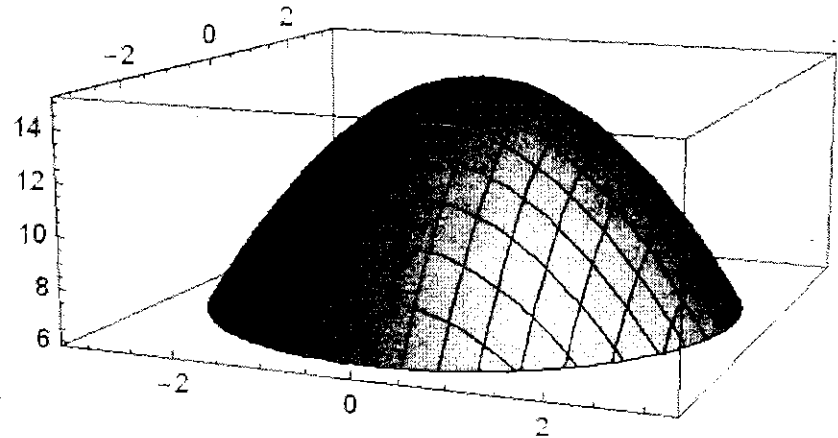
Critical pt: (0,0).

$$f_{xx} = -2, f_{yy} = -2, f_{xy} = 0$$

$$D = (-2)(-2) - (0)^2 = 4$$

$$D > 0, f_{xx} < 0, f_{yy} < 0$$

Local max!



2. $f(x,y) = x^2 + y^2$,

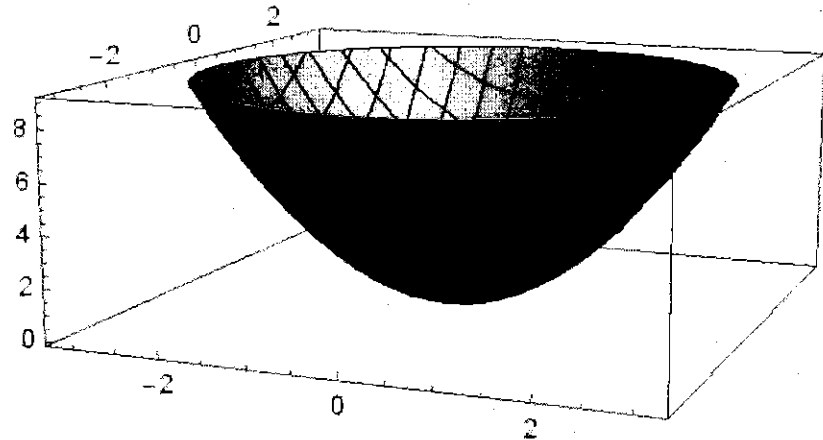
Critical pt: (0,0).

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0,$$

$$D = (2)(2) - (0)^2 = 4$$

$$D > 0, f_{xx} > 0, f_{yy} > 0$$

Local min!



3. $f(x,y) = x^2 - y^2$

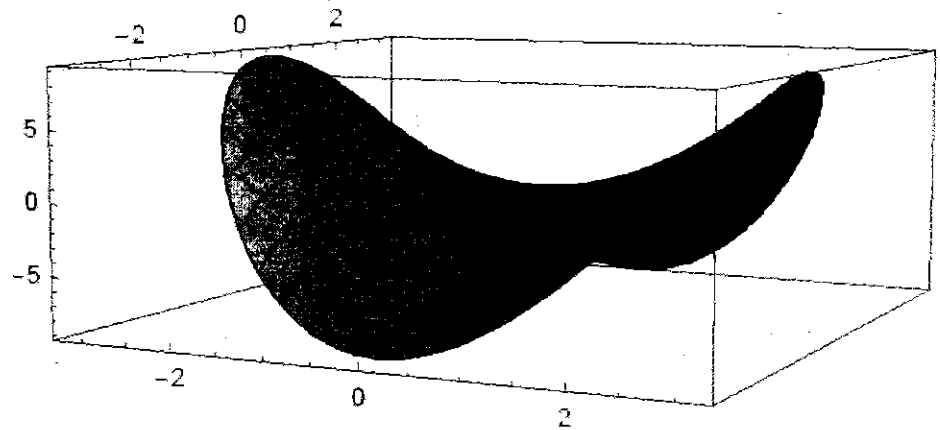
Critical pt: (0,0).

$$f_{xx} = 2, f_{yy} = -2, f_{xy} = 0,$$

$$D = (2)(-2) - (0)^2 = -4$$

$D < 0$ (note also, $f_{xx} < 0, f_{yy} > 0$)

Saddle point!



Examples from old exams:

1. Find and classify all critical points for

$$f(x, y) = x^2 + 4y - x^2y + 1$$

$$\textcircled{1} \quad 2x - 2xy \stackrel{?}{=} 0$$

$$\textcircled{2} \quad 4 - x^2 \stackrel{?}{=} 0$$

$$\textcircled{1} \Rightarrow 2x(1-y) = 0 \Rightarrow x=0 \text{ or } y=1$$

$$\textcircled{2} \Rightarrow x^2 = 4 \Rightarrow x = -2 \text{ or } x = 2$$

WHAT WORKS IN BOTH!!!

$x=0$ DOES NOT WORK IN BOTH.

$x=-2, y=1$ DOES WORK IN BOTH

$x=2, y=1$ DOES WORK IN BOTH

$(-2, 1)$	$(2, 1)$
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$$f_{xx} = 2 - 2y, \quad f_{yy} = 0, \quad f_{xy} = -2x$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 0 - 4x^2$$

AT $(-2, 1)$ $D < 0$ SADDLE POINT

AT $(2, 1)$ $D < 0$ SADDLE POINT

2. Find and classify all critical points for

$$f(x, y) = \frac{9}{x} + 3xy - y^2$$

$$\textcircled{1} \quad f_x = -\frac{9}{x^2} + 3y \stackrel{?}{=} 0$$

$$\textcircled{2} \quad f_y = 3x - 2y \stackrel{?}{=} 0$$

$$\textcircled{2} \Rightarrow 3x = 2y \Rightarrow y = \frac{3}{2}x$$

$$\textcircled{1} + \textcircled{2} \Rightarrow -\frac{9}{x^2} + 3\left(\frac{3}{2}x\right) \stackrel{?}{=} 0$$

$$-\frac{9}{x^2} + \frac{9}{2}x = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{MULT. BY } x^2$$

$$-9 + \frac{9}{2}x^3 = 0$$

$$\frac{9}{2}x^3 = 9$$

$$x^3 = 2$$

$$x = (2)^{1/3}$$

$$\Rightarrow y = \frac{3}{2}(2)^{1/3}$$

$(2^{1/3}, \frac{3}{2}(2)^{1/3})$

$$f_{xx} = \frac{18}{x^3}, \quad f_{yy} = -2, \quad f_{xy} = 3$$

$$D = \frac{18}{x^3}(-2) - (3)^2$$

For $x = 2^{1/3}$ $D < 0$

SADDLE POINT

3. Find and classify all critical points for

$$f(x, y) = x^2y - 9y - xy^2 + y^3$$

$$\text{① } f_x = 2xy - y^2 \stackrel{?}{=} 0$$

$$\text{② } f_y = x^2 - 9 - 2xy + 3y^2 \stackrel{?}{=} 0$$

$$\text{①} \Rightarrow y(2x - y) = 0$$

$$\Rightarrow y = 0 \quad \text{OR} \quad \underbrace{2x - y = 0}_{2x = y}$$

TWO CASES

$$\text{i) } y = 0 \Rightarrow \text{② } \left. \begin{array}{l} x^2 - 9 - 0 + 0 = 0 \\ \Rightarrow x = \pm 3 \end{array} \right\} \begin{array}{l} (-3, 0) \\ (3, 0) \end{array}$$

$$\text{ii) } y = 2x \Rightarrow \text{② } \left. \begin{array}{l} x^2 - 9 - 2x(2x) + 3(2x)^2 \stackrel{?}{=} 0 \\ x^2 - 9 - 4x^2 + 3 \cdot 4x^2 \stackrel{?}{=} 0 \end{array} \right\}$$

$$-9 + 9x^2 = 0$$

$$9x^2 = 9$$

$$x^2 = 1$$

$$x = \pm 1$$

$$\Rightarrow y = 2x = \pm 2$$

$$f_{xx} = 2y, \quad f_{yy} = -2x + 6y$$

$$f_{xy} = 2x - 2y$$

$$(-3, 0) \Rightarrow f_{xx} = 0, f_{yy} = 6, f_{xy} = -6,$$

$$D = (0)(6) - (-6)^2 < 0$$

SADDLE POINT

$$(3, 0) \Rightarrow f_{xx} = 0, f_{yy} = -4, f_{xy} = 6$$

$$D = (0)(-4) - (6)^2 < 0$$

SADDLE POINT

$$(-1, -2) \Rightarrow f_{xx} = -4, f_{yy} = -10$$

$$f_{xy} = 2$$

$$D = 40 - (2)^2 = 36 > 0$$

LOCAL MAX

$$(1, 2) \Rightarrow f_{xx} = 4, f_{yy} = 10$$

$$f_{xy} = -2$$

$$D = 40 - (-2)^2 = 36 > 0$$

LOCAL MIN

$$(-1, -2)$$

$$(1, 2)$$